Rotational dynamics

Rigid body: A rigid body is a solid body in which particles are compactly arranged & their positions are not disturbed by any external force applied on it. A rigid body can undergo both translational motion & rotational motion.

Translation motion: A rigid body is said to be in translation motion if every particle in it suffers equal displacement.

Rotational motion: A rigid body is said to be in rotational motion if every particle in it go[es] in concentric circles with same angular velocity but different linear velocities.

Moment of inertia.

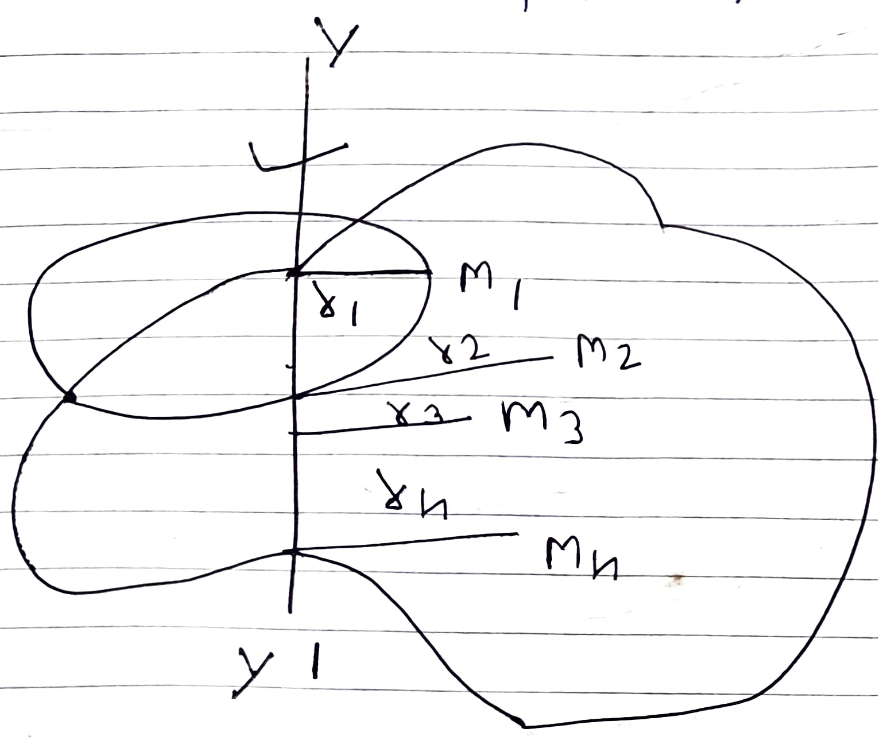

Fig: A rigid body rotating about YY'

Consider a rigid body consisting of large number of particles of masses, $m_1, m_2, m_3 \dots m_n$ at a distances of $r_1, r_2, r_3, \dots r_n$ respectively from axis YY' as shown in fig. The moment of inertia of particles of masses $m_1, m_2, m_3, \dots m_n$ about an axis YY' are $m_1 r_1^2, m_2 r_2^2, m_3 r_3^2 \dots m_n r_n^2$ respectively. The moment of inertia of whole rigid body about YY' is

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \dots + m_n r_n^2$$

$$\boxed{I = \sum_{i=1}^{n} m_i r_i^2}$$

The moment of inertia of a body about an axis is defined as the sum of product of masses of various particles & square of their perpendicular distances from the axis of rotation. It depends upon axis of rotation choosen & distribution of man of body. It's SI unit is kgm^2 & cgs unit is gcm^2. Its dimension is $[ML^2T^0]$.

Calculation of moment of inertia

(1) Moment of inertia of thin uniform rod about an axis passing through its centre & perpendicular to its length.

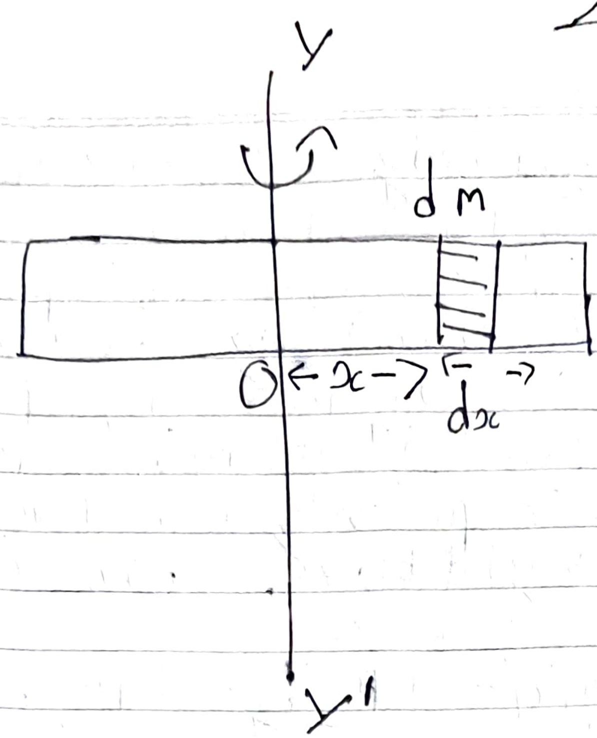

Fig: A thin uniform rod rotating about YY'

→ Let us consider a thin uniform rod of mass M & length L rotating about YY' as shown in fig. To calculate the moment of inertia of thin uniform rod. Consider a thin element of thickness dx, whose mass is dm and is at x distance from O. Then moment of inertia of this thin element about YY' is $x^2 dm$.

Mass per unit length of rod = $\frac{M}{L}$

Mass of the thin element of thickness dx is dm
$= \frac{M}{L} dx$

Moment of inertia of whole rod about YY' is

$$I = \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 \, dm$$

$L \to M$

$1 \to \frac{M}{L}$

$dm = \frac{M}{L} dx$

$$I = \left[\frac{x^3}{3}\right]_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$\frac{(1/2)^3}{3} - \frac{(-1/2)^3}{3}$$

$$\frac{1}{3}\left[\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3\right]$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 \times \frac{M}{L} dx$$

$$\frac{M}{L} \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 dx$$

$$\frac{M}{L} \times \frac{1}{3}\left[\left(\frac{L}{2}\right)^3 + \left(\frac{1}{2}\right)^3\right]$$

$$\frac{M}{3L \times 8}\left[L^3 + L^3\right]$$

$$\frac{M}{3L} \times \frac{\cancel{2}L^3}{\cancel{8}\,4}$$

$$I = \frac{ML^3}{12L} = \frac{ML^2}{12}$$

(2) (i) Moment of inertia of thin uniform rod about an axis passing through its one end & perpendicular to its length.

→ let us consider a thin uniform rod of mass M & length L rotating about YY' as shown in fig. To calculate the moment of inertia of thin uniform rod consider a thin element of thickness dx whose mass is dm & is at x distance from O. Then moment of inertia of this thin element of thickness dx whose mass is dm & is at x distance from O. Then moment of inertia of this element about YY' is $x^2 dm$.

Mass per unit length of rod $= \frac{M}{L}$

Mass of the thin element of thickness dx is $dm = \frac{M}{L}dx$

Now, moment of inertia of whole rod about YY',

$$I = \int_0^l x^2 \, dm$$

$$\int_0^l x^2 \times \frac{M}{L} \, dx$$

$$I = \frac{M}{L} \int_0^L x^2 \, dx$$

$$I = \frac{M}{L} \times \left[\frac{x^3}{3}\right]_0^L$$

$$I = \frac{M}{L} \times \frac{L^3}{3}$$

$$I = \frac{ML^3}{3L}$$
$$I = \frac{ML^2}{3}$$

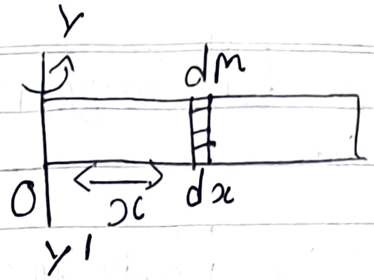

Fig: Thin uniform rod rotating about YY'

(Q) What is the diff between MI of a rod about an axis passing through its one end & through its centre ⊥ʳ to length.

$$I_a - I_B = \frac{ML^2}{3} - \frac{ML^2}{12}$$
$$\frac{ML^2}{4}$$

* Torque: Turning effect or rotating effect of force is called torque.
It is denoted by $\vec{\tau}$ & defined by
$$\vec{\tau} = \vec{r} \times \vec{F}$$
$$\tau = rF\sin\theta \ \hat{n}$$ where $\hat{n}$ is unit vector
In magnitude,
$$\tau = rF\sin\theta$$
If $\theta = 0^\circ$ & 180°
$$\tau = 0$$
If $\theta = 90^\circ$
$$\tau = rF = \text{maximum}$$

* It is a vector (Axial vector)
* It's SI unit is Nm
* It's dimensional formula is $[ML^2T^{-2}]$

H Relation betⁿ torque, moment of inertia & angular acceleration.

-> Consider a rigid body rotating about an axis YY' & is under the action of constant torque τ.

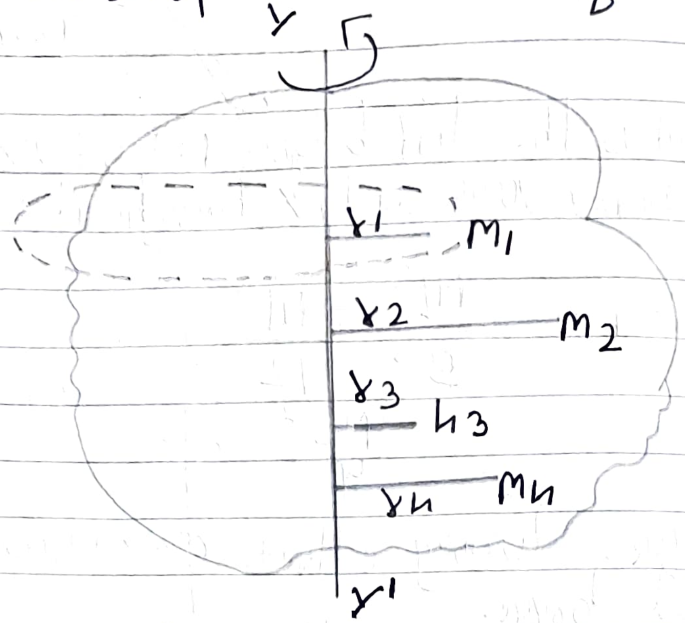

Fig:- A rigid body rotating about YY'

Suppose the rigid body consists of large number of particles of masses $M_1, M_2, M_3 \ldots M_n$ at a distances of $r_1, r_2, r_3 \ldots r_n$ respectively from axis YY'. The magnitude of ~~torque fig:- A rigid body rotating~~ acting on the particle of mass M_1 is

$$\tau_1 = r_1 F_1$$
$$= r_1 \times M_1 a_1$$
$$= r_1 \times M_1 \times \alpha r_1 \qquad a_1 = \alpha r_1, \text{ where } \alpha \text{ is angular momentum}$$
$$= M_1 r_1^2 \alpha$$

Similarly, the magnitude of torque acting on the particles of masses $M_2, M_3, \ldots M_n$ are $M_2 r_2^2 \alpha$, $M_3 r_3^2 \alpha, \ldots\ldots M_n r_n^2 \alpha$

The total torque acting on the whole rigid body is given by

$$\tau = m_1 r_1^2 \alpha + m_2 r_2^2 \alpha + m_3 r_3^2 \alpha + \ldots + m_n r_n^2 \alpha$$
$$= [m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots + m_n r_n^2] \alpha$$
$$\tau = \left[\sum_{i=1}^{n} m_i r_i^2\right] \alpha$$

$$\tau = I\alpha$$ where, $\sum_{i=1}^{n} m_i r_i^2 = I =$ moment of inertia of body about YY'.

Angular momentum:- The moment of linear momentum of an object is called angular momentum. It is denoted by L & defined by

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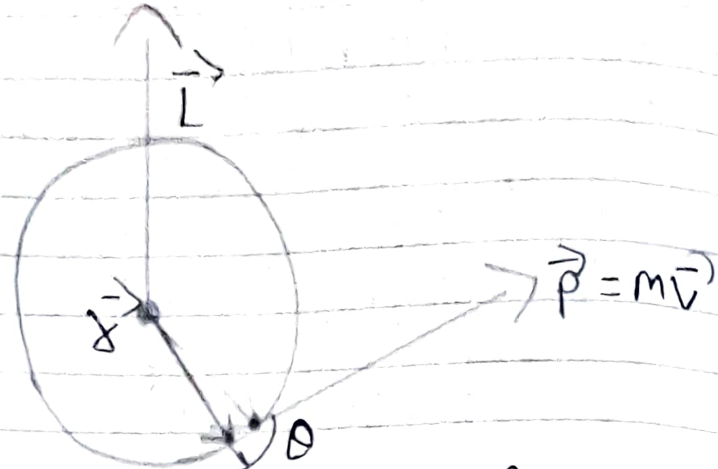
$$\vec{L} = \vec{r} \times \vec{p}$$
$$rp\sin\theta\,\hat{n}$$, where $\hat{n}$ is unit vector

In magnitude
$$L = rp\sin\theta$$
If $\theta = 0^\circ$ & 180°
$$L = 0$$
& If $\theta = 90^\circ$
$$L = rp$$
$$r \times mv$$
$\because P = mv$ (linear momentum)
$$L = mvr$$

1. It's SI unit is $kg\,m^2 s^{-1}$
2. Its dimensional formula is $[ML^2T^{-1}]$
3. It is a vector (Axial vector)



Relation betn angular momentum & moment of inertia

→ Consider a rigid body rotating about an axis YY' as shown in figure.

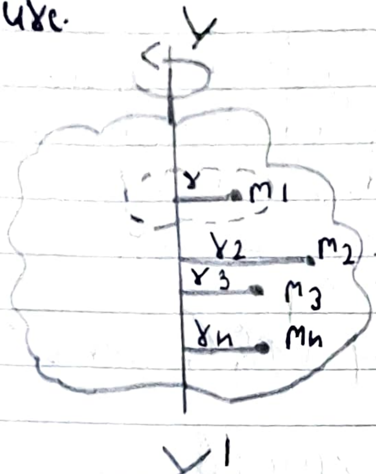


Fig: A rigid body rotating about YY'

Suppose the rigid body consists of large number of particles of masses $m_1, m_2, m_3 \dots m_n$ at a distance $r_1, r_2, r_3, \dots r_n$ respectively from axis of rotation YY'.

The magnitude of angular momentum of particle of mass m_1 is

$$L_1 = r_1 \times p_1$$
$$r_1 \times m_1 v_1$$
$$r_1 \times m_1 \times \omega r_1 \quad \because v_1 = \omega r_1 \text{ where } \omega \text{ is angular velocity}$$
$$L_1 = m_1 r_1^2 \omega$$

Similarly, the magnitude of angular momentum of particles of masses $m_2, m_3, \ldots, m_n$ are $m_2 r_2^2 \omega$, $m_3 r_3^2 \omega \ldots m_n r_n^2 \omega$ respectively.

Now, The magnitude of angular momentum of rigid body is given by

$$L = m_1 r_1^2 \omega + m_2 r_2^2 \omega + \ldots + m_n r_n^2 \omega$$
$$[m_1 r_1^2 + m_2 r_2^2 + \ldots + m_n r_n^2] \omega$$
$$\left[\sum_{i=1}^{n} m_i r_i^2\right] \omega$$

$L = I\omega$ where $\sum_{i=1}^{n} m_i r_i^2 = I = M.I$ of rigid body about YY'.

Relation betn torque & angular momentum

→ The angular momentum of a body is given by

$$L = I\omega \quad \text{— (i)}$$

where I is M.I of body about given axis & ω is angular velocity.

Diff (i) wrt 't'

$$\frac{dL}{dt} = \frac{dI\omega}{dt}$$
$$= I\frac{d\omega}{dt} + \omega\frac{dI}{dt}$$
$$I\frac{d\omega}{dt} + 0 \quad (I \text{ is constant for fixed axis})$$
$$\frac{dL}{dt} = I\alpha \quad \because \frac{d\omega}{dt} = \alpha = \text{angular acceleration}$$
$$\frac{dL}{dt} = \tau \quad \text{— (ii)} \quad \text{where } I\alpha = \tau = \text{Torque acting on a body}$$

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Eqn (ii) gives relation betn torque & angular momentum. Thus, the torque acting on a rigid body is defined as the rate of change of angular momentum of body.

Law of conservation of angular momentum:- "If the system is isolated then the total angular momentum of the system always remains constant."

i.e $L =$ constant.

Proof:- The torque acting on a rigid body is defined as the rate of change of angular momentum of body. i.e.

$$\tau = \frac{dL}{dt}$$

If the system is isolated ($\tau = 0$) then,

$$0 = \frac{dL}{dt}$$

On integrating above eqn, we get

$$L = \text{constant}$$

$$I\omega = \text{constant}$$

In general

$$I_1\omega_1 = I_2\omega_2$$

which is law of conservation of angular momentum.

Note: Moment of inertia less that's why spinning rate is high when the earth is near the sun but angular momentum remains constant.

Work Done by a Couple:

→ Two equal & opposite parallel forces acting on a rigid body such that their lines of action donot coincide form a couple.

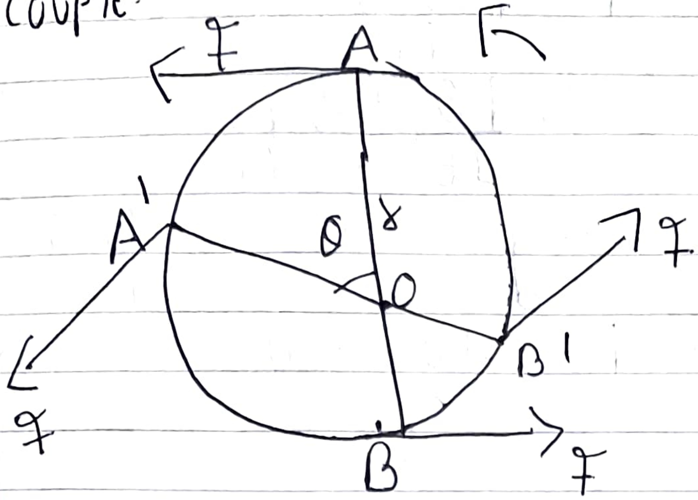


Fig: Work Done by a couple on rotating a wheel.

Let us consider a wheel of radius 'r' rotating about its center 'O' as shown in figure. Suppose two equal & opposite parallel forces 'F' acts tangentially at point A & B.

Work Done by each force = Force × displacement

$= Fs$

$= Fr\theta$

$= rF\theta$ $\quad$ $(s = r\theta$ where θ = angular displacement$)$

Total work done $\quad W = rF\theta + rF\theta$

$= 2rF\theta$

$W = \tau\theta$

~~$W = Fs$~~

~~$W = \tau\theta$~~

$2rF = \tau$ = Torque due to couple.

Work done by Couple $= rF\theta \Rightarrow \tau\theta$

Power: Rate of doing work with respect to time is called power.

$$P = \frac{dW}{dt}$$

$$P = \frac{d(\tau\theta)}{dt}$$

$$\tau \frac{d\theta}{dt} + \theta \frac{d\tau}{dt}$$

$$\tau \frac{d\theta}{dt} + 0 \quad (\tau \text{ is constant})$$

$$P = \tau\omega \quad \text{—} \ (1)$$

$$\frac{d\theta}{dt} = \omega$$

Ang vel

Rotational kinetic Energy

Kinetic Energy of rotating body is called rotational kinetic Energy.

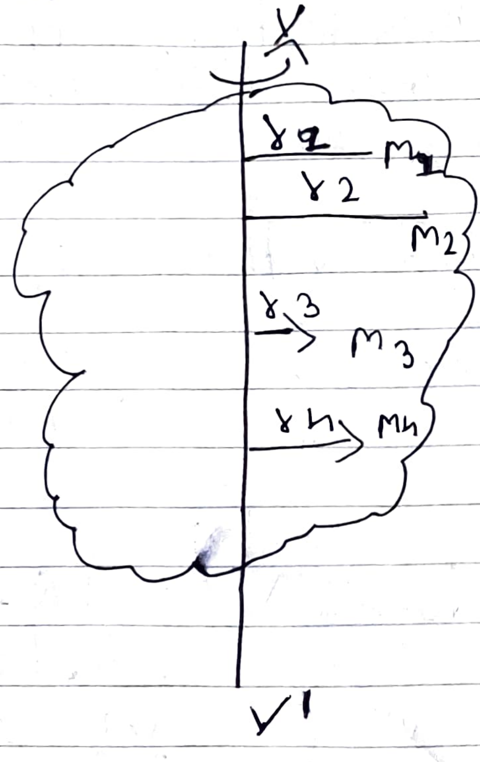


Fig: A rigid body rotating about YY' radius

Let us consider a rigid body rotating about an axis YY' as shown in figure. Suppose the rigid body consist of large number of particles of masses $m_1, m_2, m_3, \ldots m_n$ at a distances of $r_1, r_2, r_3, \ldots r_n$ respectively from axis YY'.

The rotational kinetic energy of mass

$$m_1 = \frac{1}{2} m_1 v_1^2$$

$$= \frac{1}{2} m_1 (r_1 \omega)^2 \quad \because v_1 = r_1 \omega$$

$$\frac{1}{2} m_1 r_1^2 \omega^2, \text{ where } \omega = \text{angular velocity}$$

Similarly, the rotational kinetic energy of particle of masses $m_2, m_3, \ldots m_n$ are $\frac{1}{2} m_2 r_2^2 \omega^2$,

$\frac{1}{2} m_3 r_3^2 \omega^2, \ldots \frac{1}{2} m_n r_n^2 \omega^2$ resp.

Now,

Rotational kinetic energy of rigid body

$$\frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2 + \ldots + \frac{1}{2} m_n r_n^2 \omega^2$$

$$\frac{1}{2} \left[m_1 r_1^2 + m_2 r_2^2 + \ldots + m_n r_n^2 \right] \omega^2$$

$$\frac{1}{2} \left[\sum_{i=1}^{n} m_i r_i^2 \right] \omega^2$$

$$\text{Rotational K.E} = \frac{1}{2} I \omega^2$$

$$\because \sum_{i=1}^{n} m_i r_i^2 = I = \text{M.I of body about } YY'$$

Kinetic Energy of rolling body

$x = 2\pi R$

Fig. wheel rolling on a flat horizontal surface

Let us consider a wheel of mass 'm' & radius 'R' rolling on a horizontal surface as shown in fig. When a body rolls it rotates about its centre of mass & undergoes displacement in forward direction. So the body passes both translational & rotational motion.

Translation kinetic Energy $(E_{trans}) = \frac{1}{2} mv^2$

& rotational $(E_{rot}) = \frac{1}{2} I\omega^2$

Total kinetic Energy of rolling body $= E_{trans} + E_{rot}$

$$= \frac{1}{2} mv^2 + \frac{1}{2} I\omega^2 \quad \text{— (i)}$$

k = radius of gyration

$$= \frac{1}{2} mv^2 + \frac{1}{2} mk^2\omega^2$$

$$= \frac{1}{2} mv^2 + \frac{1}{2} mk^2\frac{v^2}{R^2}$$

$$E = \frac{1}{2} mv^2 \left[1 + \frac{k^2}{R^2} \right] \quad \text{— (ii)}$$

$\omega = \frac{v}{R}$

Acceleration of rolling body down the inclined plan

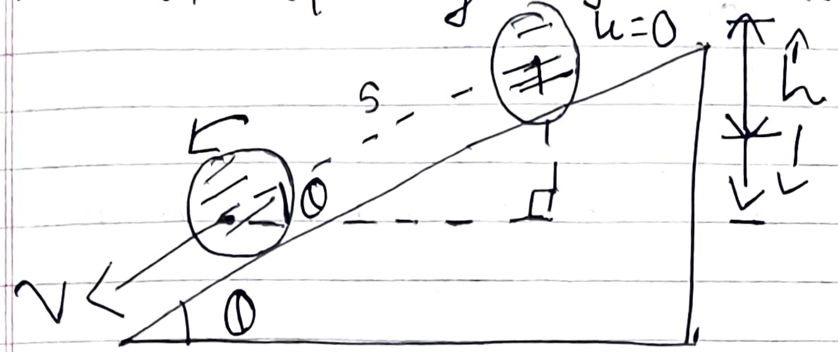


Fig:- A sphere rolling down in the inclined plane

Let us consider a sphere of mass 'm' & radius R rolling down in a inclined plane without slipping as shown in figure.

As no slipping occurs, the mechanical energy is conserved

i.e loss in P.E = Gain in K.E

$$mgh = \frac{1}{2}mv^2\left[1+\frac{k^2}{R^2}\right]$$

$$2gh = v^2\left[1+\frac{k^2}{R^2}\right]$$

$$v^2 = \frac{2gh}{\left(1+\frac{k^2}{R^2}\right)}$$

$$\left[\sin\theta = \frac{h}{s}\right]$$

$$v^2 = \frac{2gs\sin\theta}{\left[1+\frac{k^2}{R^2}\right]} \quad \text{— (i)}$$

From eqn of motion $v^2 = u^2 + 2as$

$v^2 = 0^2 + 2as$ $(u=0)$

$v^2 = 2as$ — (ii)

From (i) & (ii)

$$2as = \frac{2gs\sin\theta}{\left(1+\frac{k^2}{R^2}\right)}$$

$$a = \frac{g \sin\theta}{\left[1 + \frac{k^2}{R^2}\right]} \quad \text{(iii)}$$

Equation of translational motion & Eqn of rotational Motion

translational	rotational
(1) $s = ut$	$\theta = \omega_0 t$
(2) $v = u + at$	$\omega = \omega_0 + \alpha t$
(3) $v^2 = u^2 + 2as$	$\omega^2 = \omega_0^2 + 2\alpha\theta$
(4) $s = ut + \frac{1}{2}at^2$	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

where

ω_0 is initial angular velocity,
ω is final angular velocity
α is angular acceleration
θ is angular displacement

Theorem of parallel axis

→ The moment of inertia of a rigid body about an axis is equal to the moment of inertia about parallel axis passing through the center of mass plus the product of mass of the body & square of the distance between the two axis.

i.e :-

$$I = I_{cm} + Mr^2$$

cm = center of mass

$V = u + at$

$rw = rw_0 + r\alpha t$

$rw = r(w_0 + \alpha t)$

$w = w_0 + \alpha t$

Proof:

$$I_{AB} = \Sigma m(r+x)^2$$
$$\Sigma m(r^2 + x^2 + 2rx)$$
$$= \Sigma mr^2 + \Sigma mx^2 + 2r\,\Sigma mx \quad (\Sigma mx \to 0)$$
$$I_{AB} = Mr^2 + I_{CD}$$

[Σmx = Sum of moment of ~~inertia~~ particles about center of mass = 0]

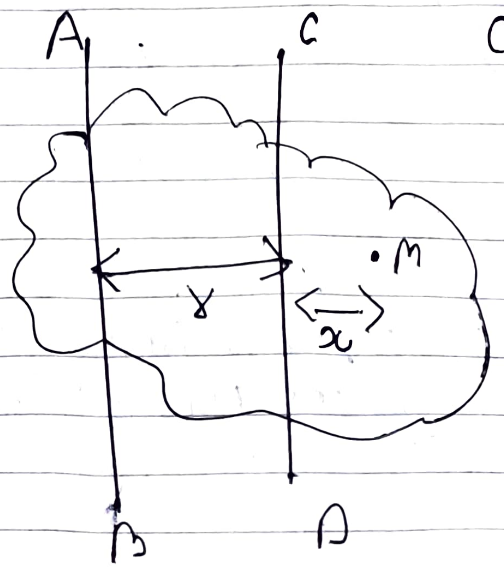


Fig: Theorem of parallel axis

Theorem of ⊥r axis

→ It states that moment of inertia about an axis ⊥r a laminar object is equal to the sum of moments of inertia about 2 mutually ⊥r axis within the plane of the object & intersecting at the same point.

i.e. $I_x + I_y = I_z$

Proof:

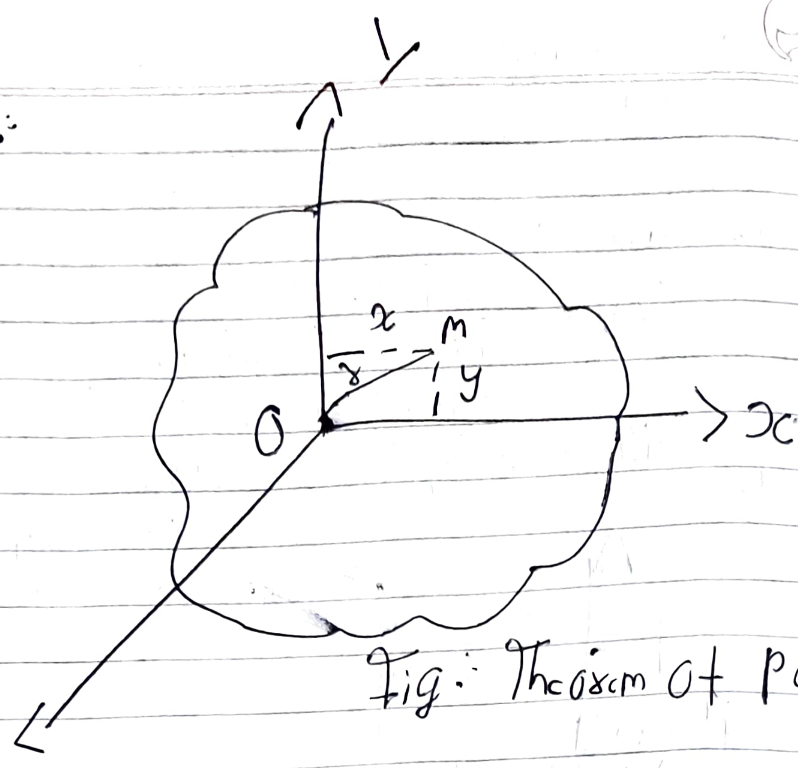


Fig: Theorem of perpendicular axis

The moment of inertia of lamina about z axis is $I_z = \sum m r^2$. Similarly, the moment of inertia of lamina about x-axis is $I_x = \sum m y^2$ & about y-axis is $I_y = \sum m x^2$.

Then

$$I_x + I_y = \sum m y^2 + \sum m x^2$$

$$\sum m (y^2 + x^2)$$

$$\sum m r^2 \qquad (y^2 + x^2 = r^2)$$

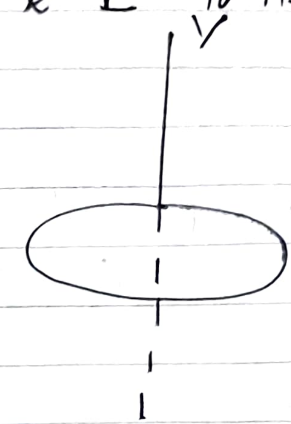
$$I_x + I_y = I_z$$

Moment of Inertia of diff rigid body

① Thin uniform circular ring.

ⓐ About an axis through its centre & $\perp^r$ to its plane is

$I = MR^2$

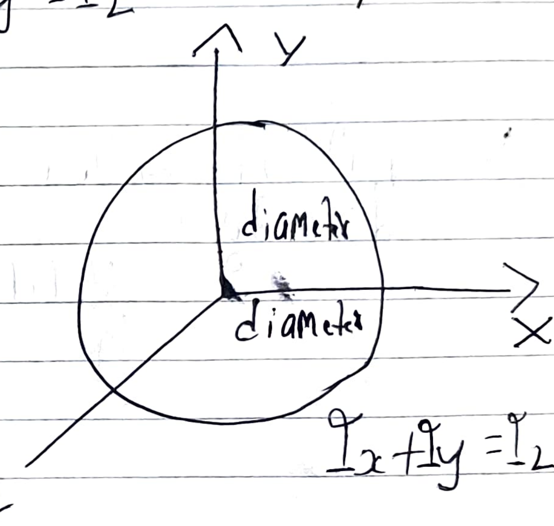


ⓑ About its diameter

$I + I = MR^2$ $\quad \therefore I_x + I_y = I_z$

$2I = MR^2$

$I = \frac{MR^2}{2}$



② Thin uniform circular disc:

ⓐ About an axis through its centre & $\perp^r$ to its plane is $I = \frac{MR^2}{2}$

(b) About its diameter

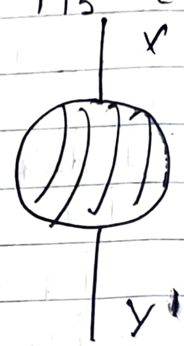
$$I + I = \frac{MR^2}{2} \qquad [I_x + I_y = I_z]$$

$$2I = \frac{MR^2}{2}$$

$$I = \frac{MR^2}{4}$$

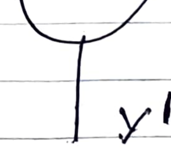
(3) MI of solid sphere about its diameter (own axis)

$$I = \frac{2}{5} MR^2$$



(4) MI of walled hollow sphere (spherical shell) about its diameter is

$$I = \frac{2}{3} MR^2$$



~~Radial~~ Radius of Gyration:

→ Defined as the distance from axis of rotation to a point where total mass of body is supposed to be concentrated, so the moment of inertia about the axis may remains same.

It is denoted by 'k'. In terms of radius of gyration. The moment of inertia of a rigid body is given by

$I = Mk^2$ — (1)

Suppose a rigid body consist of large no of particles masses $m_1, m_2, m_3, \ldots m_n$. let distances $r_1, r_2, r_3, \ldots r_n$ resp from the axis of rotation Moment of inertia of body is given by

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots + m_n r_n^2$$

$$I = m(r_1^2 + r_2^2 + r_3^2 + \ldots r_n^2)$$

$$= \frac{mn(r_1^2 + r_2^2 + \ldots + r_n^2)}{n}$$

Since $M = mn$

$$I = \frac{M(r_1^2 + r_2^2 + \ldots + r_n^2)}{n}$$

$$Mk^2 = M\left(\frac{r_1^2 + r_2^2 + r_3^2 + \ldots + r_n^2}{n}\right)$$

$$k = \sqrt{\frac{r_1^2 + r_2^2 + r_3^2 + \ldots + r_n^2}{n}}$$

k = root mean sqr distance of various particles

Calculation of Radius of Gyration.

(1) Thin uniform rod.

→ The MI of thin uniform rod about its centre & ⊥r to its length is

$$I = \frac{Ml^2}{12} \quad \text{— (1)}$$

MI of body in terms of radius of gyration is

$$I = Mk^2 \quad \text{— (ii)}$$

from (1) & (ii)

$$Mk^2 = \frac{Ml^2}{12}$$

$$k = \frac{l}{\sqrt{12}}$$

(ii) Radius of Gyration of a solid sphere.

= M.I of solid sphere about its axis is given by

$$I = \frac{2}{5}MR^2 \quad \text{— (i)}$$

M.I of body in terms of radius of gyrations,

$$I = Mk^2 \quad \text{— (ii)}$$

$$\cancel{M}k^2 = \frac{2}{5}\cancel{M}R^2$$

$$k = \sqrt{\frac{2}{5}}\, R$$