Mechanics Periodic Motion

Periodic Motion

→ If motion of object repeats after fixed interval of time then it is called periodic motion.

Eg:- Motion of Earth round the sun, Motion of Simple pendulum etc.

Simple Harmonic Motion (SHM):

→ Simple harmonic motion is a periodic motion in which restoring force is directly proportional to the displacement of body from its mean position. The restoring force is always directed towards mean position.

In SHM

Restoring force (F) $\propto$ displacement (y)

$$F = -ky$$

where k is force constant & -ve sign indicates restoring force is opp of displacement of a body.

Eqn of Displacement, Velocity & Acceleration in SHM

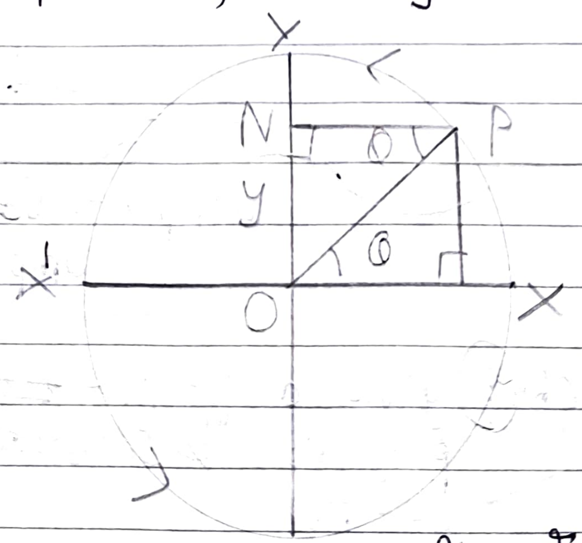


Fig:- Foot of ⊥r N executing SHM on axis YOY'

$V = r\omega$ (tangential velocity on the circle (circular motion))

$V = r\omega \cos \omega t$ (velocity of the projection of the circular motion)

Consider a particle moving in a circle of radius 'r' with uniform speed in anti clockwise direction as shown in fig.

In time 't' let the particle is at point P & θ be the angular displacement. Let M & N be the foot of $\perp^r$ drawn from P on axis XOX' & YOY' resp. The displacement of foot N on axis YOY' is ON.

In ΔONP

$$\sin\theta = \frac{ON}{OP}$$

$$\sin\theta = \frac{y}{r}$$

$$y = \sin\theta \, r$$

$$y = r \sin \omega t \quad -(1)$$

↳ A is amplitude which is equivalent to the radius 'r'.

In ΔOMP

$$\cos\theta = \frac{OM}{OP}$$

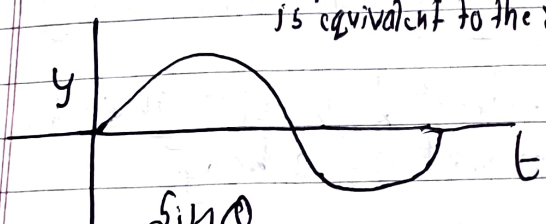
$$\cos\theta = \frac{x}{r}$$

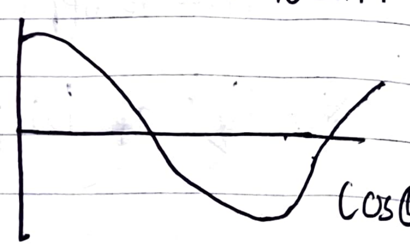
$$x = r\cos\theta$$

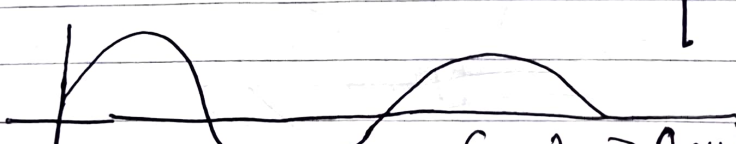
$$x = r\cos\omega t$$

$$\therefore \boxed{\omega = \frac{\theta}{t}}$$

↳ Connects circular motion to SHM

* 



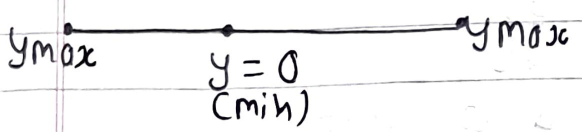
* 

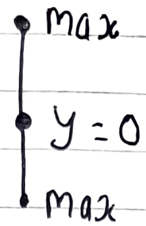
Simple → Oscillation with constant amplitude (single frequency)

Eqn (1) gives displacement ~~eqn~~ eqn of SHM which is periodic & sinusoidal (sine or cosine) fn of time.

$$\boxed{V = \pm \omega \sqrt{r^2 - x^2}}$$

* Harmonic :- Sinusoidal   Particle which show SHM : Harmonic Oscillator





Velocity :- The rate of change of displacement of an object with respect to time is called velocity.

i.e. $V = \dfrac{dy}{dt}$

$$\frac{d\, r\sin\omega t}{dt} \qquad (y = r\sin\omega t)$$

$$r\,\frac{d\sin\omega t}{dt}$$

$$V = r\cdot\cos\omega t\cdot\omega$$

$$V = r\omega\cos\omega t \quad \text{— } (2)$$

$$V = r\omega\sqrt{1-\sin^2\omega t}$$

$$V = r\omega\sqrt{\left(1-\frac{y^2}{r^2}\right)}$$

$$V = \omega\sqrt{r^2-y^2} \quad \text{— } (3)$$

Eqn (2) & (3) gives velocity eqn in S.H.M

* Special ~~eqn~~ Case :

(1) when $y = 0$ (mean position)

$$V = \omega\sqrt{r^2}$$

$$V = \omega r \text{ (max)}$$

(2) when $y = r$ (Extreme position)

$$V = \omega\sqrt{r^2 - r^2}$$

$$V = 0 \quad \text{minimum}$$

Acceleration: The rate of change of velocity of an obj with time is called accl^n.

i.e: $a = \frac{dv}{dt}$

$$a = \frac{d\, r\omega \cos\omega t}{dt}$$

$$a = r\omega \frac{d \cos\omega t}{dt}$$

$$a = -r\omega \sin\omega t \cdot \omega$$

$$a = -\omega^2 r \sin\omega t \quad — (4)$$

$$a = -\omega^2 y \quad — (5)$$

$$a \propto y$$

Eqn (4) & (5) gives expr^{sn} for acc in S.H.M

From (5) it is clear that the acc of a particle in S.H.M is directly proportional to the displacement which is the main characteristics of SHM

* Special case

(i) when $y = 0$

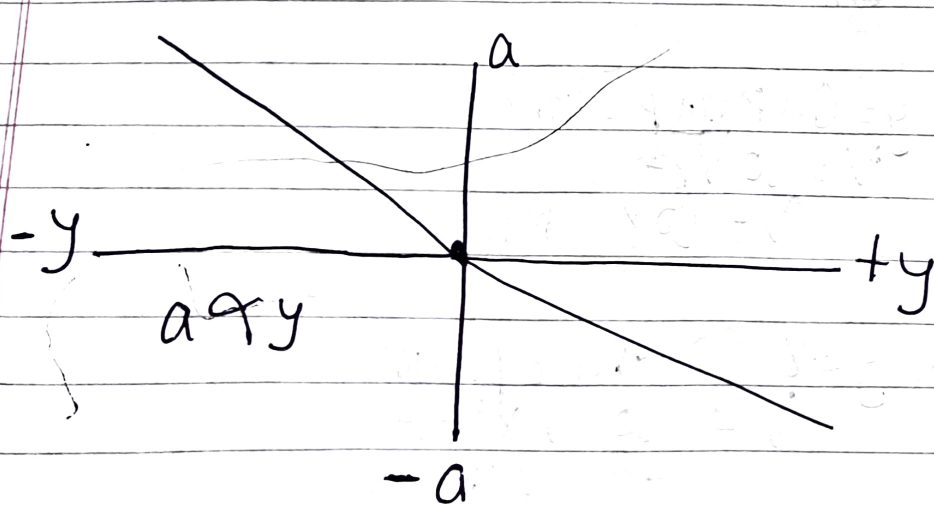
$$a = 0 \text{ (min)}$$

(ii) when $y = r$

$$|a| = \omega^2 r \quad \text{Max}$$

[For numericals take $|a|$

$$\underline{a = -\omega^2 r}$$



At any instant (t)

Amplitude: The maximum displacement of particle from its mean position in SHM is called amplitude. OR displacement amplitude.

The eqn of displacement in SHM is

$$y = r \sin \omega t$$

If $\sin \omega t = 1$

then $y_{max} = r$ = amplitude

Time period: The time taken by an object to complete 1 oscillation in S.H.M. is called its time period (T)

The eq^n of acc^n in SHM is

$$a = \omega^2 y$$

$$a = \left(\frac{2\pi}{T}\right)^2 y \qquad \left(\omega = \frac{2\pi}{T}\right)$$

$$a = \frac{4\pi^2}{T^2} y$$

$$T^2 = 4\pi^2 \frac{y}{a}$$

$$\text{Time Period } (T) = 2\pi\sqrt{\frac{y}{a}} \quad - (6)$$

Frequency (f): The number of oscillation completed per second by an object in SHM is called its frequency. It is denoted by f & given by

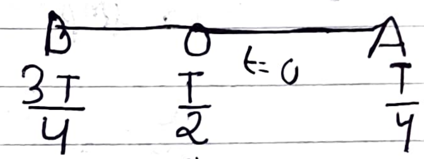
$$f = \frac{1}{T}$$

$$= \frac{1}{2\pi\sqrt{\frac{y}{a}}}$$

$$f = \frac{1}{2\pi} \times \sqrt{\frac{a}{y}} \quad (7)$$

One oscillation:

$OA \rightarrow AO \rightarrow OB \rightarrow BO$



$f = 0.5\,Hz$

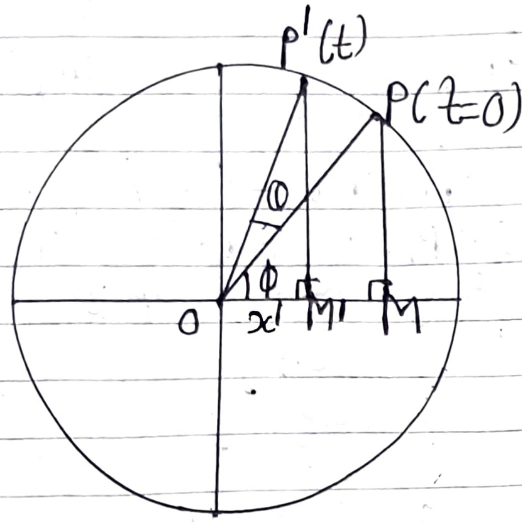
$f = \frac{1}{2} s^{-1}$

$1s = 0.5$

Phase diff betwn Vely in shm a xe
$\frac{\pi}{2}$

Phase: The phase of an oscillatory particle at any instant gives the position & direction of motion of particle with respect to its mean position. It is measured in terms of fraction of time period or fraction of angle 2π after crossing the mean position.

*



In $\Delta OM'P'$

$$\cos(\theta+\phi) = \frac{x'}{r}$$

$$x' = r\cos(\theta+\phi)$$

Actual formula for x. $[x' = r\cos(\omega t+\phi)]$

$$\text{Phase} = \omega t + \phi$$

$$\phi = \text{Initial Phase}$$

Q) In the eqn $y = 20\sin 10\pi t$, y is in mm & t is in second. Find amplitude, time period & velocity at $t=0$

$$y = 20\sin 10\pi t \quad \text{— (i)}$$

The eqn of displacement in SHM is

$$y = r\sin\omega t \quad \text{— (ii)}$$

Comparing (i) & (ii)

$$r = 20\,mm = 20\times10^{-3}\,m$$

$$\omega = 10\pi$$

$$\frac{2\pi}{T} = 10\pi$$

angular frequency - ω in shm
frequency - f

$T = \frac{1}{5} = 0.2s$

Next part, Velocity at $t=0$

$V = \omega r$

$\frac{2\pi}{T} \cdot r$

$V = \left[\frac{2\pi}{0.2} \times 20\times10^{-3}\right] = \frac{1\pi}{5}$ m/s

Graphical representation of displacement, velocity & acc^n for a particle in SHM

→ The eqn of displacement, velocity & acc^n in SHM are,

$y = r\sin\omega t$ — (i)

$v = r\omega\cos\omega t$ — (ii) → $r\omega\sin\left(\omega t + \frac{\pi}{2}\right)$

$a = -\omega^2 y$ — (iii)

(1) When $t=0$

$y = 0, a = 0$

$V = r\omega\sin\frac{\pi}{2} = r\omega$

(2) when $t = \frac{T}{4}$

$y = r\sin\frac{2\pi}{T} \cdot \frac{T}{4} = r$

$v = 0$

$a = -\omega^2 r$

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③ when $t = \frac{T}{2}$

$y = r\sin\frac{2\pi}{T}\cdot\frac{T}{2} = 0$

$v = r\omega\sin\left(\frac{2\pi}{T}\cdot\frac{T}{2} + \frac{\pi}{2}\right) = -r\omega$

$a = 0$

④ When $t = \frac{3T}{4}$

$y = r\sin\frac{2\pi}{T}\cdot\frac{3T}{4} = -r$

$v = 0$

$a = \omega^2 r$

⑤ When $t = T$

$y = r\sin\frac{2\pi}{T}\cdot T = 0$

$v = r\omega$

$a = 0$

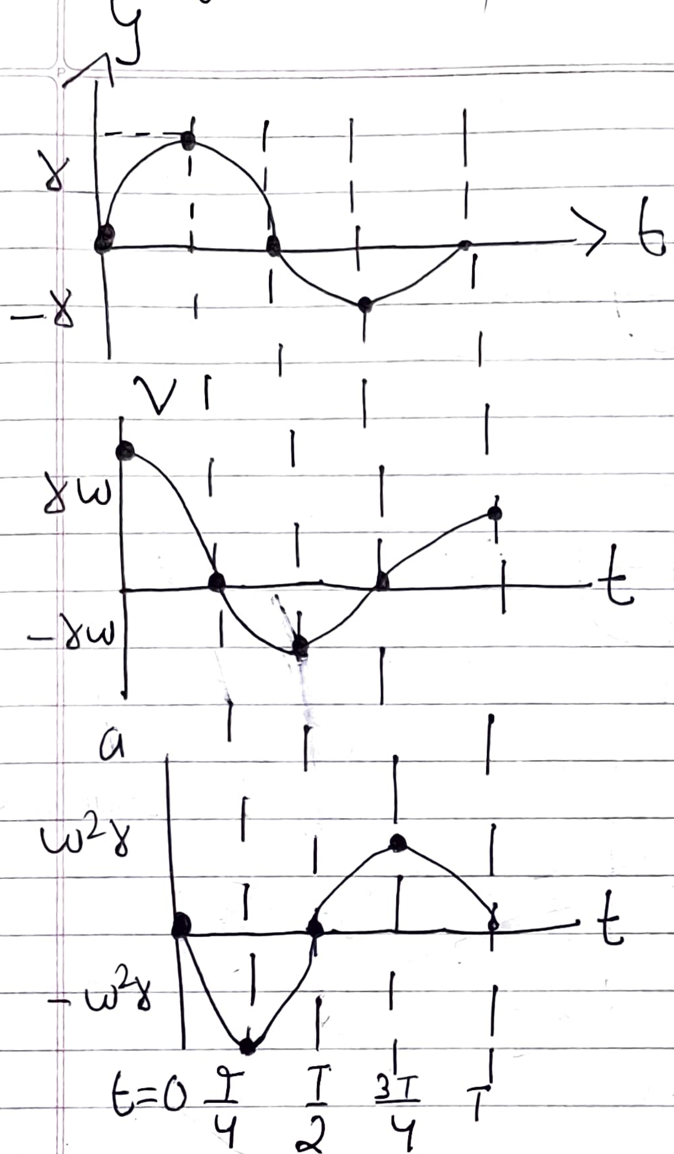
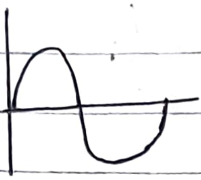


Fig: Graph showing the variation of displacement, velocity & acc^n of a particle in SHM.

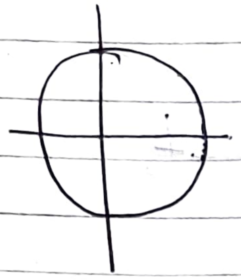
Graph in SHM

y vs t
v vs t
a vs t



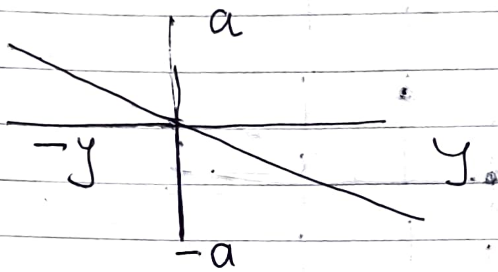
Sine Curve

v vs y
a vs v



Ellipse

a vs y



Straight line

VII Simple Pendulum

⇒ A simple Pendulum is a heavy point mass object suspended by an inextensible, weightless string from a rigid support & which is free to oscillate in a vertical plane.

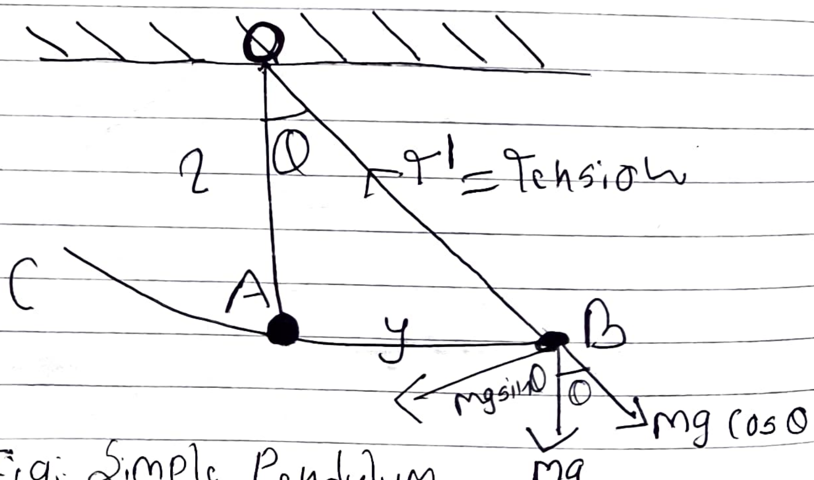


Fig: Simple Pendulum

Let m be the mass of simple Pendulum whose effective length (Distance between point of suspension & C.G of bob) is L. When the bob is at point B the weight mg can be resolved into two components $mg\cos\theta$ & $mg\sin\theta$. The component $mg\sin\theta$ acts as restoring force.

i.e Restoring force $(F) = -mg\sin\theta$ $\quad \because F = ma$

$$ma = -mg\sin\theta$$

$$a = -g\sin\theta \quad \text{— (I)}$$

For small angle,

$$\sin\theta \approx \theta = \frac{AB}{OA} = \frac{y}{L} \quad \text{— (ii)}$$

From eqn (I) & (II)

$$a = -g \cdot \frac{y}{L}$$

$$a = -\left(\frac{g}{L}\right)y \quad \text{— iii}$$

Since $\frac{g}{L}$ is constant

$$\boxed{\text{So} \quad a \propto y}$$

Hence, the motion of simple pendulum is simple harmonic.

Time period : The action of acc^n in SHM is

$$a = -\omega^2 y \quad \text{— (iv)}$$

Comparing eqn (3) with (4)

$$-\left(\frac{g}{L}\right)y = -\omega^2 y$$

$$\frac{g}{l} = \omega^2$$

$$\frac{g}{L} = \left(\frac{2\pi}{T}\right)^2 \qquad \omega = \frac{2\pi}{T}$$

$$\frac{g}{L} = \frac{4\pi^2}{T^2}$$

$$T^2 = 4\pi^2 \frac{l}{g}$$

$$\text{Time period } (T) = 2\pi\sqrt{\frac{l}{g}} \quad - (v)$$

<u>Frequency</u>

$$f = \frac{1}{T} = \frac{1}{2\pi\sqrt{\frac{l}{g}}}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \quad - (vi)$$

Second's pendulum

→ A simple pendulum having time period two second is called second's pendulum.

The time period of simple pendulum is

$$T = 2\pi\sqrt{\frac{l}{g}}$$

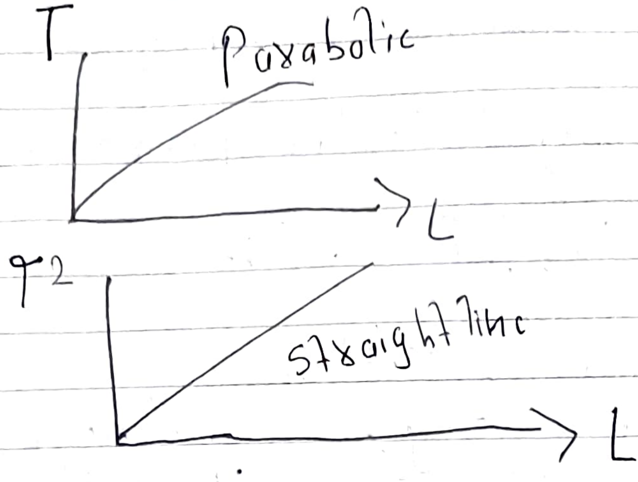
Sqr on both sides

$$T^2 = 4\pi^2 \frac{l}{g}$$

$$2^2 = 4\pi^2 \frac{l}{9.8}$$

$$l \approx 0.993\text{m}$$

T = 2 sec for Seconds' Pendulum



Drawbacks of simple pendulum:

1. Heavy point mass object & weightless string is not practically possible
2. The air resistance & buoyancies affect the motion of bob
3. The formula $T = 2\pi\sqrt{\frac{L}{g}}$ is valid only for small amplitude of oscillation.
4. The motion of bob is not strictly linear (It has rotational motion also)

Horizontal Mass - Spring system:

Consider an elastic spring of negligible mass whose one end is fixed at wall & an object of mass m is attached at another end as shown in fig.

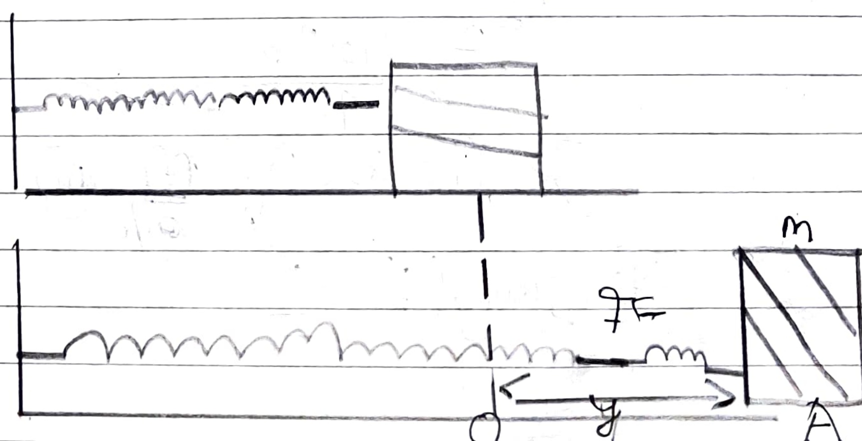


Fig: Horizontal mass spring system

If y be the elongation in the spring then from hooke's law.

restoring force (F) $\propto$ Elongation (y)

$$F = -ky$$

where k is force constant or

$$Ma = -ky \qquad \text{Since } F = ma$$

$$a = -\left(\frac{k}{m}\right)y \quad -(i)$$

Here $\frac{k}{M}$ is constant so $a \propto y$

Hence, the motion of horizontal mass spring in SHM

Time period: The eqn of acc in SHM is

$$a = \omega^2 y \quad -(ii)$$

From (i) & (ii)

$$-\left(\frac{k}{m}\right)y = -\omega^2 y$$

$$\frac{k}{m} = \omega^2$$

$$\frac{k}{m} = \left(\frac{2\pi}{T}\right)^2$$

$$\frac{k}{m} = \frac{4\pi^2}{T^2}$$

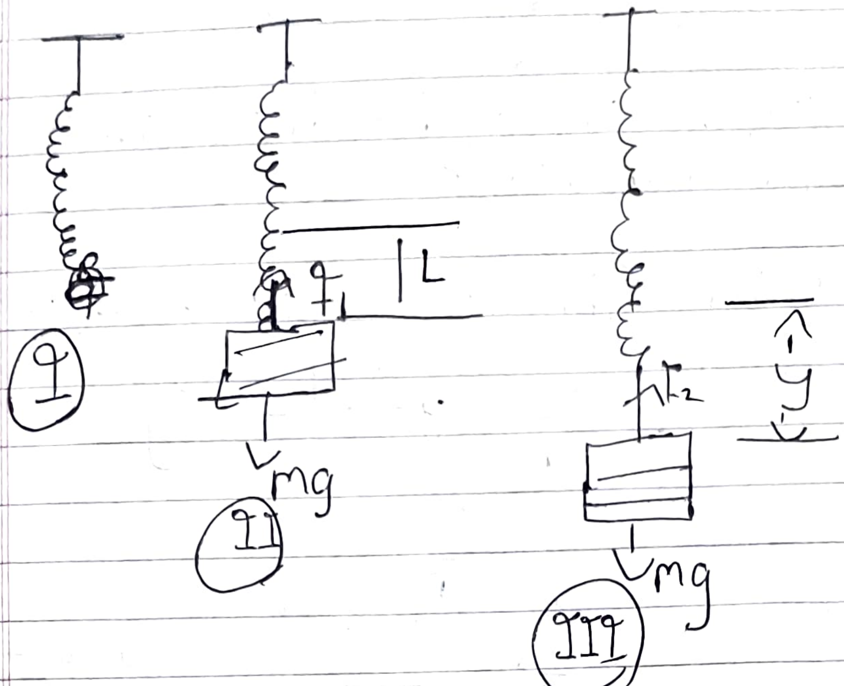
$$T^2 = \frac{4\pi^2 m}{k}$$

$$T = 2\pi\sqrt{\frac{M}{k}} \quad -(iii)$$

Frequency (f) $= \frac{1}{T}$

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \quad \text{(iv)}$$

Vertical mass - spring system



Consider a elastic spring whose one end is fixed at celling as shown in fig ① & an object of mass m is attached at other end as shown in fig ②. When the load m is connected on spring there will be elongation L then from hooke's law

Restoring force $(F_1) \propto l$

$$F_1 = -kl$$

where k is force constant. When the load is pulled down further distance y as in fig ③ then restoring force F_2.

$$F_2 = -k(l+y)$$

Now the effective restoring force

$$F = F_2 - F_1$$

$$= -k(l+y) + kl$$

$$-kl - ky + kl$$

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$$ma = -ky$$

$$a = -\left(\frac{k}{m}\right)y \quad - (i)$$

Since $\frac{k}{m}$ is constant

So $a \propto y$

Thus the motion loaded helical vertical spring is simple harmonic.

Time Period: The eqn of acc^n in SHM is

$$a = \omega^2 y \quad - (II)$$

$$-\left(\frac{k}{m}\right)y = -\omega^2 y$$

$$\frac{k}{m} = \omega^2$$

$$\frac{k}{m} = \frac{4\pi^2}{T^2} \qquad \omega = \frac{2\pi}{T}$$

$$T^2 = \frac{4\pi^2 m}{k}$$

$$T = 2\pi\sqrt{\frac{m}{k}} \quad - (iii)$$

Frequency $(f) = \frac{1}{T}$

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$F_1 \propto l$

$mg = F_1 = -kl$

$g \propto l$

$F_2 \propto (l+y)$

$F_2 = -k(l+y)$

$F = F_2 - F_1$

$-k(l+y) + kl$

$-\cancel{kl} - ky + \cancel{kl}$

$F = -ky$

$ma = -ky$

$a \propto y$

Energy in SHM

As the restoring force acts in SHM, the particle executing SHM possesses both kinetic & potential energy

① Kinetic energy (E_k): The kinetic energy of a particle of mass m moving with velocity v is given by

$$E_k = \frac{1}{2}mv^2$$

$$\frac{1}{2}m\left[\omega\sqrt{r^2-y^2}\right]^2 \qquad (v = \omega\sqrt{r^2-y^2})$$

$$E_k = \frac{1}{2}m\omega^2(r^2-y^2) \quad \text{—(i)}$$

$$\frac{1}{2}mv^2$$

$$\frac{1}{2}m(r\omega\cos\omega t)^2$$

$$\frac{1}{2}mr^2\omega^2\cos^2\omega t$$

② Potential energy (E_p):
Consider a particle executing SHM with amplitude r. The eq^n of accn SHM is

$$a = -\omega^2 y$$

where ω is angular frequency & y is displacement.

$x = r\cos\omega t$

$v = -r\omega\sin\omega t$

$E_k = \frac{1}{2}mr^2\omega^2 \sin\omega t$

The restoring force is then given by

$$F = ma$$
$$= m(-\omega^2 y)$$
$$= -m\omega^2 y$$
$$F = -ky$$

where $k = m\omega^2 =$ constant.

$w = F.s$

$\boxed{dw = F\,dy}$

$F = -ma$

$-m\omega^2 y$

$$E_k = \frac{1}{2}m\omega^2[r^2-y^2]$$

$$\frac{1}{2}mr^2\omega^2\cos^2\omega t$$

$$\frac{1}{2}mr^2\omega^2\sin\omega^2 t$$

The small work done on displacing the particle through a small dis dy against the force.

$$dw = -F\,dy$$
$$= -(-ky)\,dy$$
$$dw = ky\,dy$$

Then total work done on displacing the particle through a distance y against the force is

$$W = \int_0^y dw = \int_0^y ky\,dy = k\int_0^y y\,dy$$
$$= k\left[\frac{y^2}{2}\right]_0^y$$
$$= \frac{1}{2}ky^2$$
$$W = \frac{1}{2}m\omega^2 y \qquad \because k = m\omega^2$$

This workdone remains in the form of potential energy.

$$E_p = \frac{1}{2}m\omega^2 y^2 \quad \text{—(ii)}$$

Now, Total energy (E) = kinetic (E_k) + Potential energy

$$\frac{1}{2}m\omega^2(r^2 - y^2) + \frac{1}{2}m\omega^2 y$$
$$\frac{1}{2}m\omega^2 r^2 - \frac{1}{2}m\omega^2 y^2 + \frac{1}{2}m\omega^2 y$$
$$E = \frac{1}{2}m\omega^2 r^2 \quad \text{—(iii)}$$

Eqn III gives expression for total energy of a particle in SHM

* Special cases.

(1) When $y = 0$ (Mean position)

$$E_p = \frac{1}{2} m\omega^2 \cdot 0^2 = 0$$

$$E_k = \frac{1}{2} m\omega^2 [r^2 - 0^2] = \frac{1}{2} m\omega^2 r^2$$

Maximum = Total energy

(2) when $y = r$ (Extreme position)

$$E_p = \frac{1}{2} m\omega^2 r^2 = \text{Max} = \text{Total energy}$$

& $E_k = \frac{1}{2} m\omega^2 (r^2 - r^2) = 0$

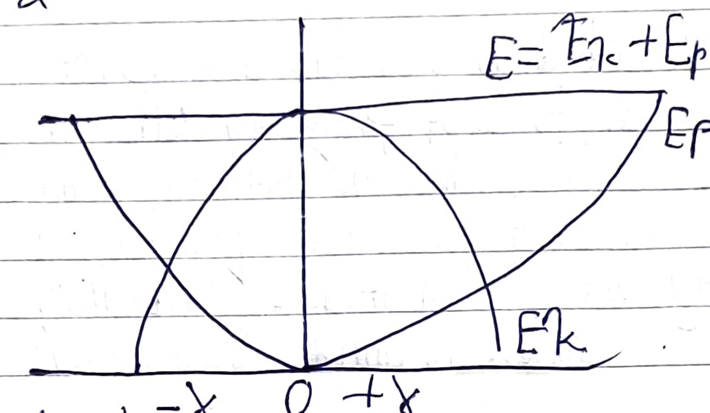

Fig: Graph showing the variation of kinetic energy of a particle as function of position

Free Oscillation / Undamped Oscillation

→ When a body is capable for oscillation is given some initial displacement from its equilibrium position & left free, it begins to oscillate with its own frequency with constant amplitude. This oscillation is called free oscillation.

Eg: Oscillation of simple pendulum in Vaccum.

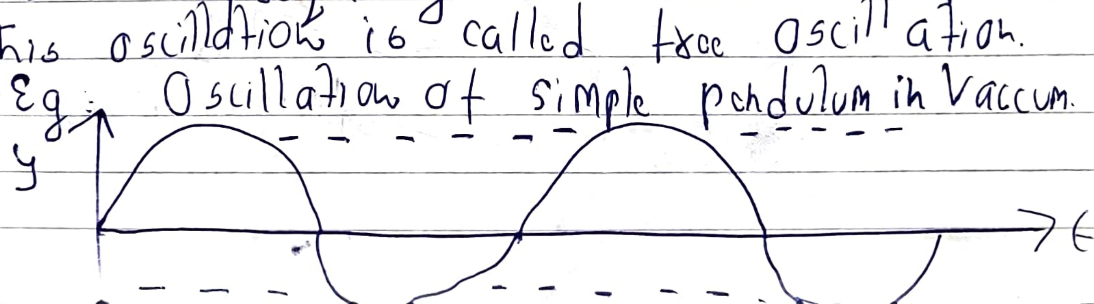

Fig: Free oscillation

Damped oscillation/damping: The oscillation in which amplitude of oscillating particle goes on decreasing with time is called damped oscillation or damping.

Eg: Oscillation of simple pendulum in liquid

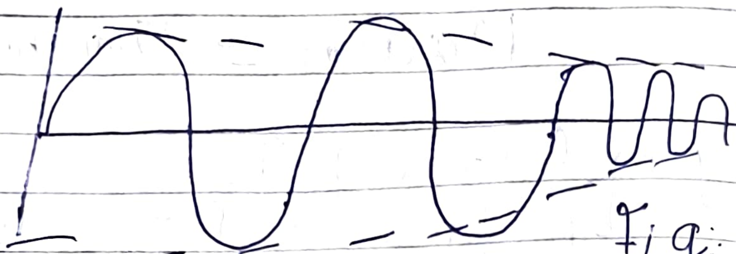

Fig: Damped oscillation

Forced oscillation & resonance:

→ When a body is maintained in a state of oscillation by an external force of frequency other than natural frequency of body. Then the oscillation is called forced oscillation. If applied frequency is equal to natural frequency of body then amplitude of vibration becomes maximum. This condition is called resonance/resonate vibration.

Angular SHM

→ Angular SHM is an oscillatory motion in which restoring torque is directly proportional to the angular displacement of the body & is always directed towards mean position/ ~~mean~~ equilibrium.

i.e

$$\tau \propto \theta$$
$$\tau = -k\theta$$

where k is torsion constant.

$I\alpha = -k\theta$ $\qquad$ [$\tau = I\theta$ where I is moment of inertia & α is angular acc^n]

$\alpha = -\left(\frac{k}{I}\right)\theta$ — (I)

alpha

Since $\frac{k}{I}$ is constant

Angular acc^n (α) $\propto$ Angular displacement (θ)

Time period: The eqn of angular acc^n in SHM is

$$\alpha = -\omega^2\theta \quad \text{— (II)}$$

from (I) & (II)

$$-\left(\frac{k}{I}\right)\theta = -\omega^2\theta$$

$$\frac{k}{I} = \omega^2$$

$$\frac{k}{I} = \left(\frac{2\pi}{T}\right)^2$$

$$\frac{k}{I} = \frac{4\pi^2}{T^2}$$

$$T^2 = \frac{4\pi^2 I}{k}$$

$$T = 2\pi\sqrt{\frac{I}{k}} \quad \text{— iii}$$

Frequency: $f = \frac{1}{T}$

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{I}} \quad \text{— iv}$$

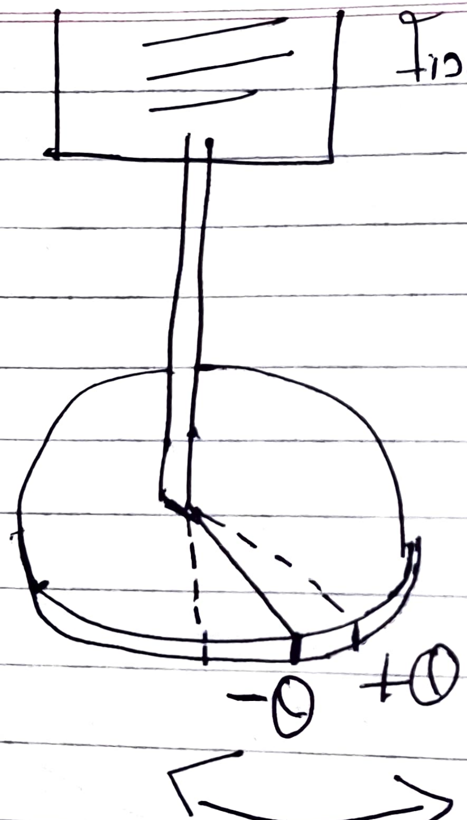

Fig: Torsonal pendulum